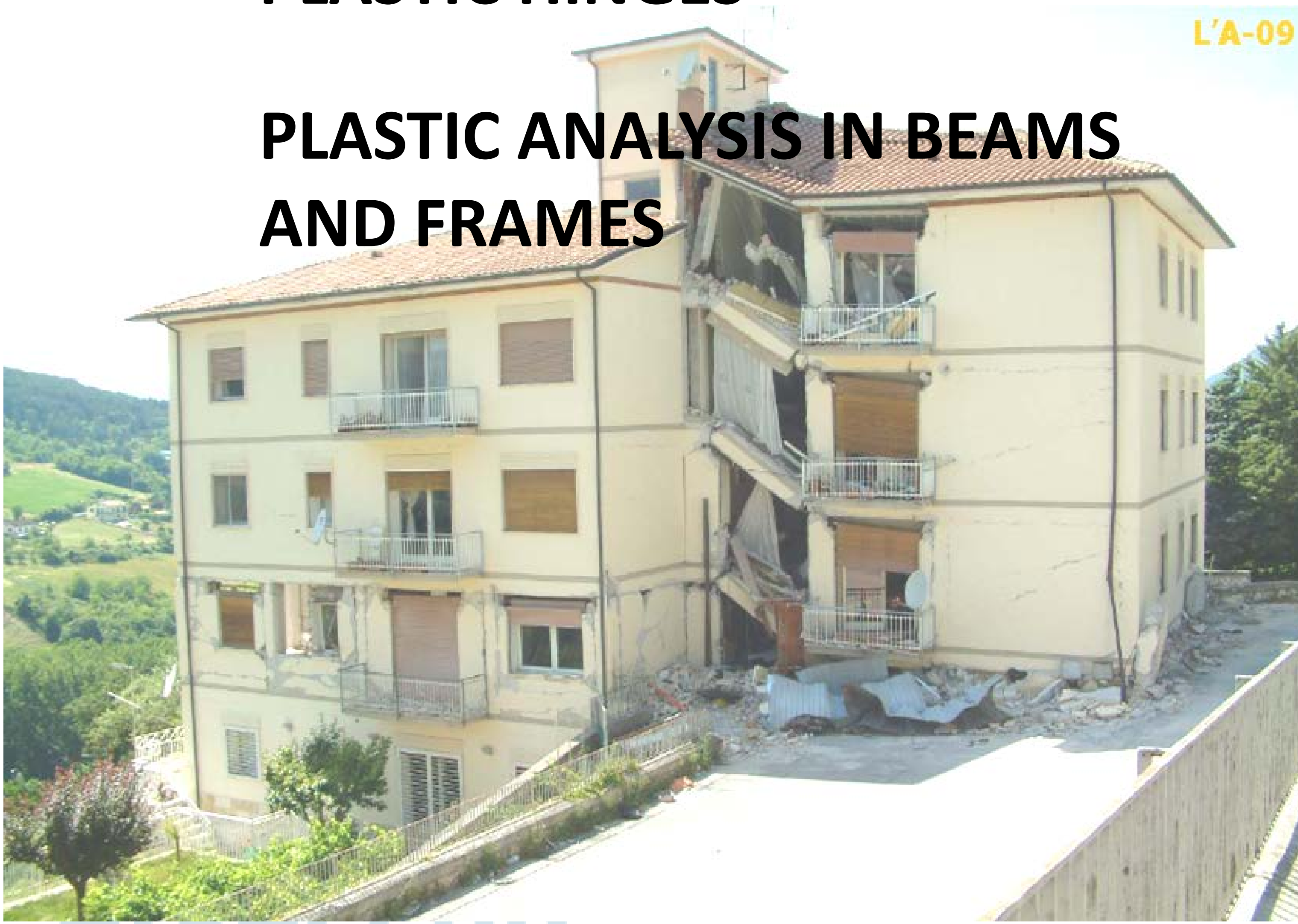
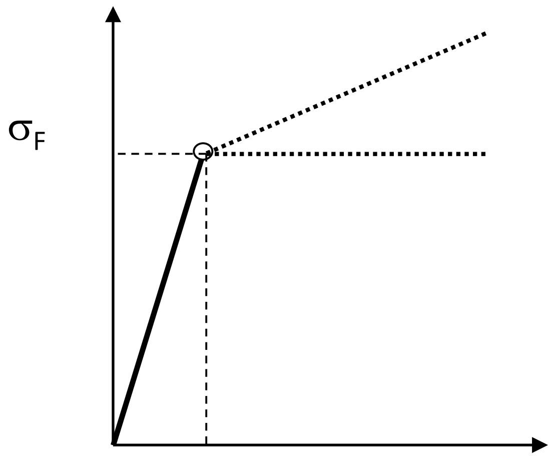
RESISTÈNCIA DE MATERIALS I ESTRUCTURES A L'ENGINYERIA
UNIVERSITAT POLITÈCNICA DE CATALUNYA

# PLASTIC HINGES

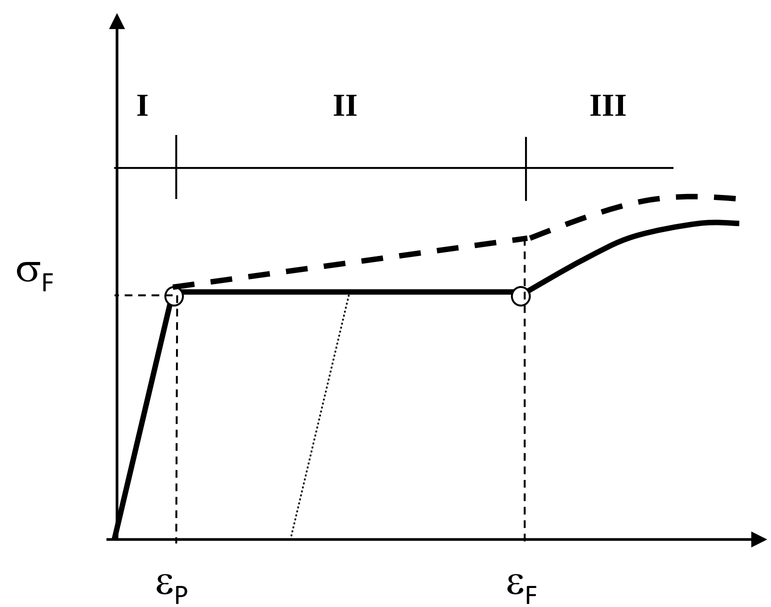
# PLASTIC ANALYSIS IN BEAMS AND FRAMES

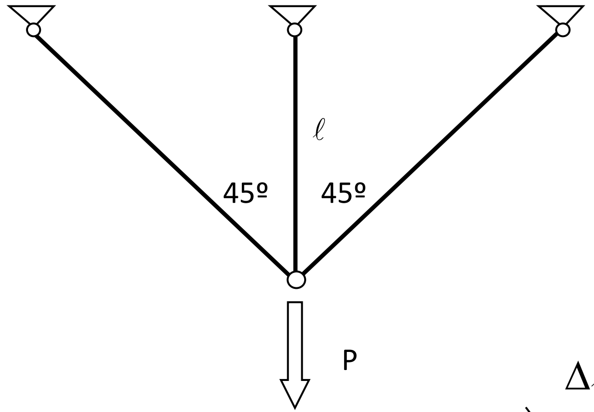


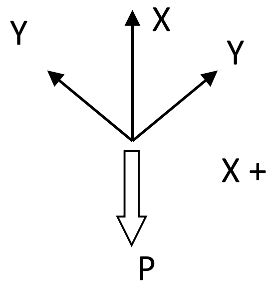



When in a given section the limite stress is reached, the structure collapses

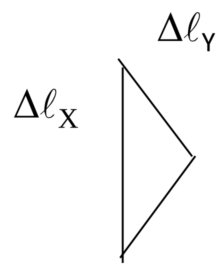
| | |
|---|---|
| Region I | Elastic region |
| Region II | Plastic Region |
| Region III | Large deformations |






$$X + 2Y\cos 45 = P$$




$$\Delta\ell_X \cos 45 = \Delta\ell_Y$$

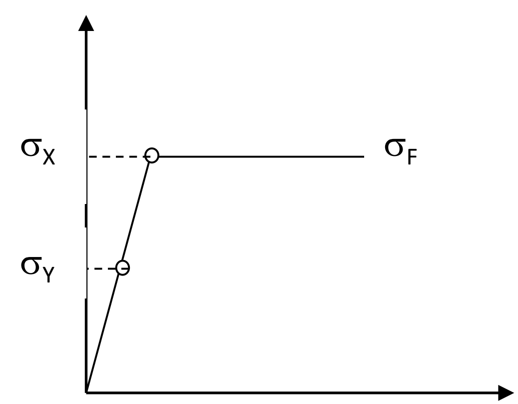
$$\frac{X}{EA}\,\ell\,\frac{\sqrt{2}}{2} = \frac{Y}{EA}\,\ell\sqrt{2}$$

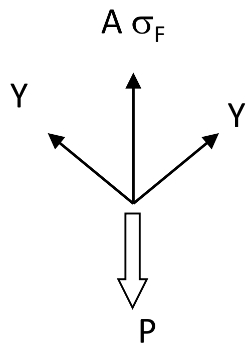
$$X = \frac{2P}{2+\sqrt{2}} \qquad Y = \frac{P}{2+\sqrt{2}}$$

**X= 2Y**

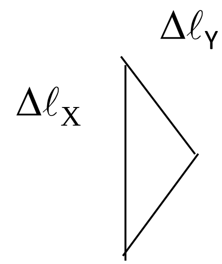
$$\sigma_X = \frac{X}{A} = \frac{P}{A}\frac{2}{2+\sqrt{2}} = \sigma_F$$

$$P_{LE} = \frac{2+\sqrt{2}}{2} A\,\sigma_F \quad \Rightarrow \quad \sigma_X = \sigma_F \quad \sigma_Y = \frac{\sigma_F}{2}$$






$$A\,\sigma_F + 2\,Y\cos 45 = P$$

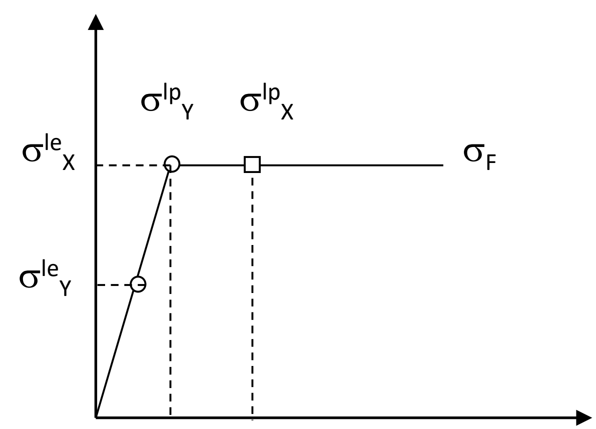



$$\Delta\ell_X \cos 45 = \Delta\ell_Y$$

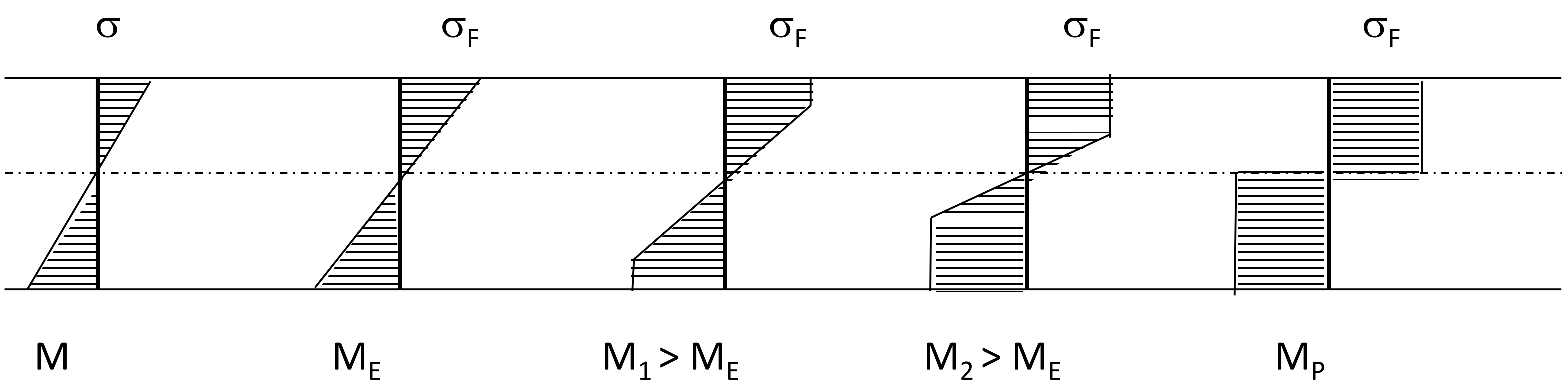
$$Y = \frac{P - A\sigma_F}{\sqrt{2}} = \frac{\sqrt{2}}{2}\left(P - A\sigma_F\right)$$

$$\frac{\sqrt{2}}{2}\left(P - A\sigma_F\right) = A\sigma_F$$

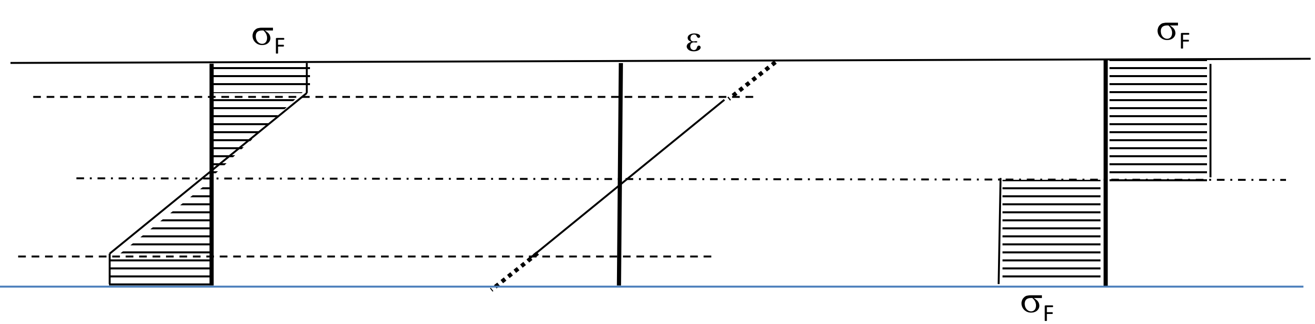
$$P_{LP} = \frac{2}{\sqrt{2}}\left(1 + \frac{\sqrt{2}}{2}\right)A\sigma_F = \frac{2+\sqrt{2}}{\sqrt{2}}\,A\sigma_F = \sqrt{2}\;P_{LE}$$



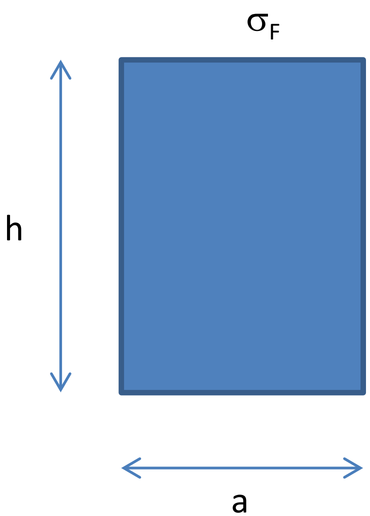

$$\sigma_X = \sigma_Y = \sigma_F$$




$$\sigma = \frac{M}{I}\, y \qquad \varepsilon = \frac{\sigma}{E} = \frac{M}{EI}\, y$$

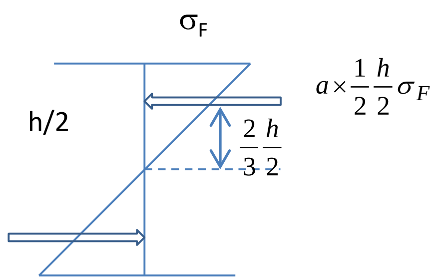


# Materials Resistance

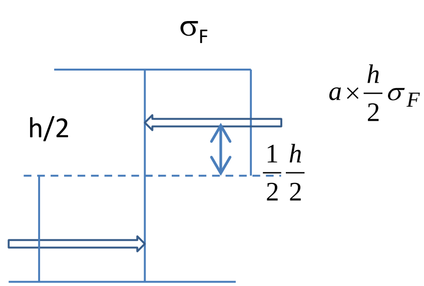



$$\sigma = \frac{M\,y}{I} \qquad y = \frac{h}{2} \qquad I = \frac{1}{12}a\,h^3 \qquad M = \frac{a\,h^2}{6}\sigma$$

$$M_{le} = \frac{a\,h^2}{6}\sigma_F$$

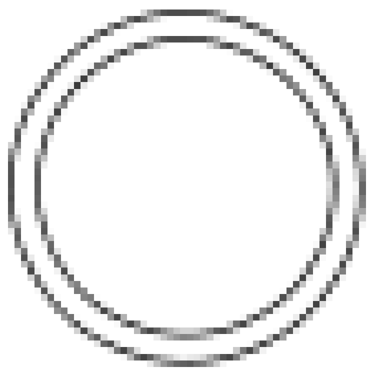



$$M_{le} = 2 \times a\frac{1}{2}\frac{h}{2}\sigma_F \times \frac{2}{3}\frac{h}{2} = \frac{a\,h^2}{6}\sigma_F$$

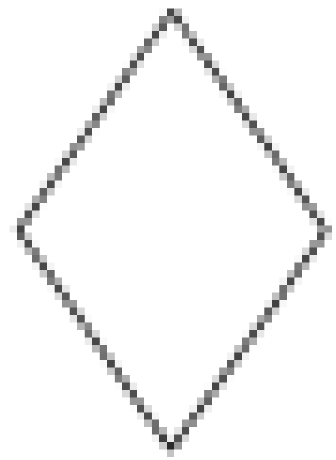



$$M_{lp} = 2 \times a\frac{h}{2}\sigma_F \times \frac{1}{2}\frac{h}{2} = \frac{a\,h^2}{4}\sigma_F$$

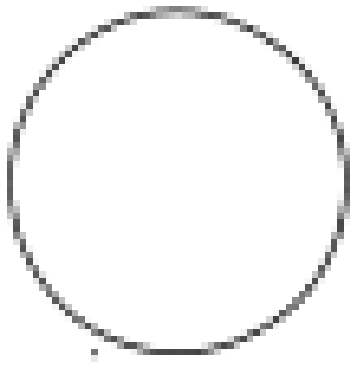
$M_{lp} / M_{le} = 1,5$ Shape Factor



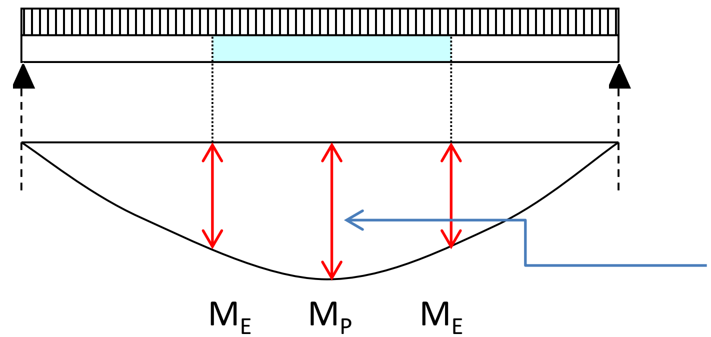
$M_P / M_E = 1,27$



$M_P / M_E = 2$



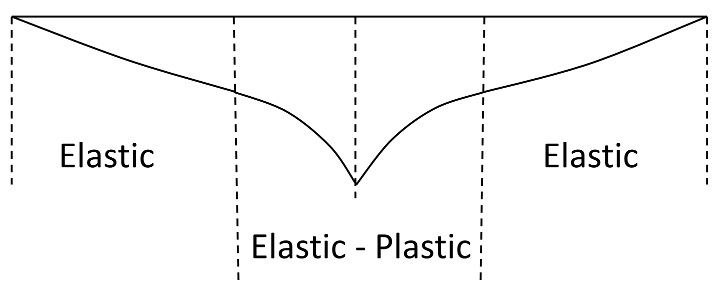
$M_P / M_E = 1,7$

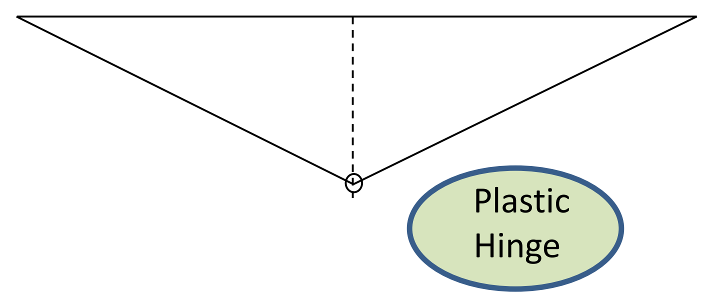


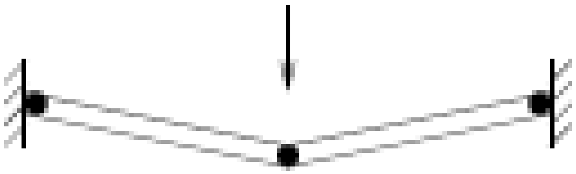

$$M_f = \frac{q\ell^2}{8}$$

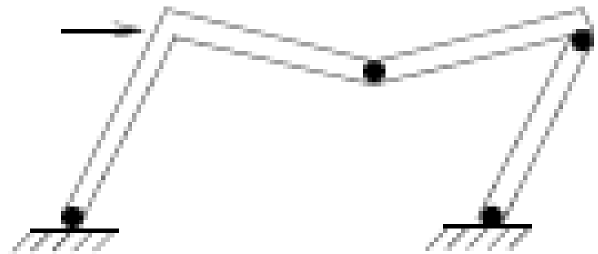
$$q_{LE} = \frac{8M_E}{\ell^2} \qquad q_{LP} = \frac{8M_P}{\ell^2}$$

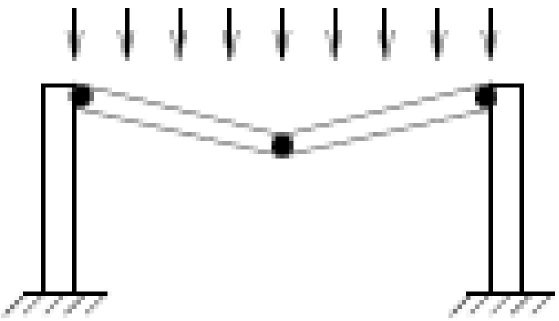
$$\frac{q_{LP}}{q_E} = \frac{\frac{8M_P}{\ell^2}}{\frac{8M_E}{\ell^2}} = \frac{M_P}{M_E} = f$$

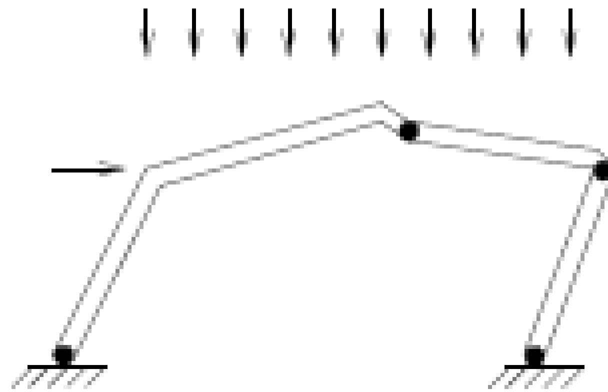


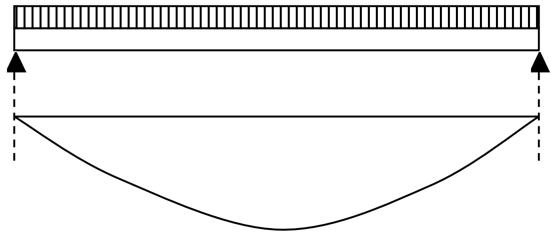









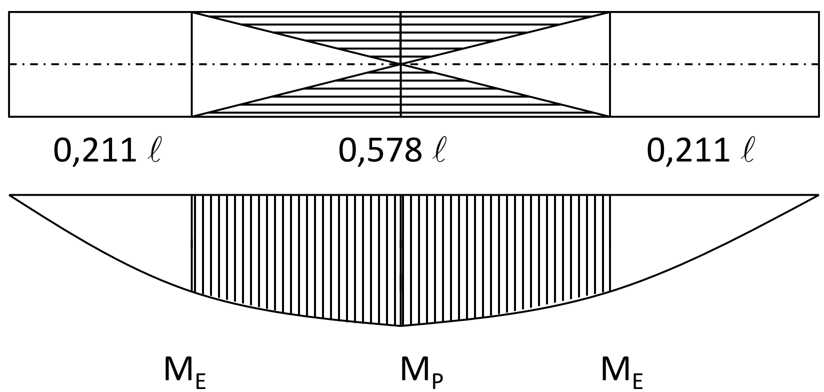


$$M_f = \frac{q_{LP}\ell}{2}x - \frac{q_{LP}x^2}{2} = \frac{4M_P}{\ell}x - \frac{4M_P}{\ell^2}x^2$$

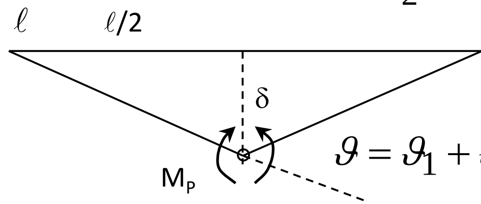
$$M_E = \frac{4M_P}{\ell}x - \frac{4M_P}{\ell^2}x^2 \qquad M_E = M_P/f$$

$$\ell^2 = 4\,f\,\ell\,x - 4\,f\,x^2 \quad \text{for a rectangular section: } f = 1{,}5 \quad 6\,x^2 - 6\,\ell\,x + \ell^2 = 0$$

$$x = 0{,}211\,\ell \qquad x = 0{,}789\,\ell$$



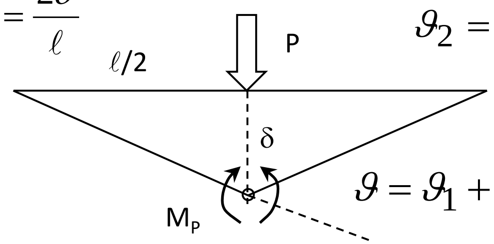
$$\vartheta_1 = \frac{2\delta}{\ell} \qquad \vartheta_2 = \frac{2\delta}{\ell}$$




$$\vartheta = \vartheta_1 + \vartheta_2 = \frac{4\delta}{\ell}$$

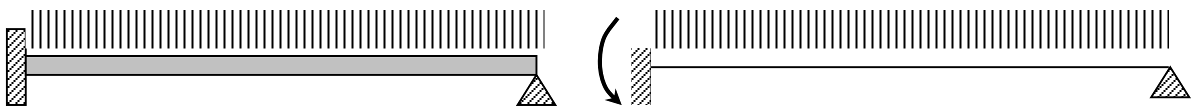
$$q_{LP}\left(\frac{1}{2}\,\ell\,\delta\right) = M_P\,\frac{4\delta}{\ell} \qquad \Rightarrow \qquad q_{LP} = \frac{8M_P}{\ell^2}$$

$$\vartheta_1 = \frac{2\delta}{\ell} \qquad \vartheta_2 = \frac{2\delta}{\ell}$$

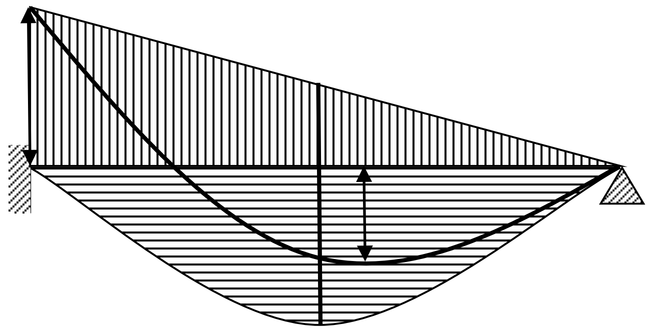



$$\vartheta = \vartheta_1 + \vartheta_2 = \frac{4\delta}{\ell}$$

$$P_{LP}(\delta) = M_P\,\frac{4\delta}{\ell} \qquad \Rightarrow \qquad P_{LP} = \frac{4M_P}{\ell} \quad \text{which coincides with} \quad M_P = \frac{P}{2}\,\frac{\ell}{2}$$



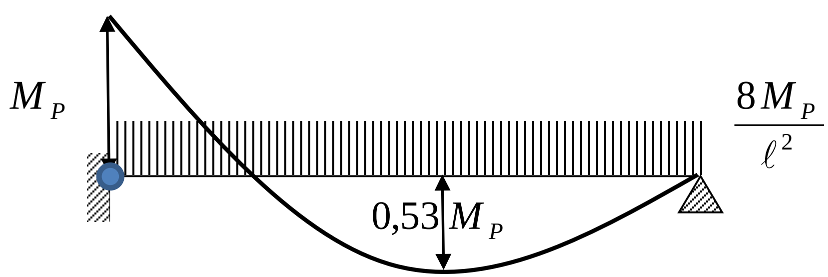
$$M = \frac{q\ell^2}{8}$$



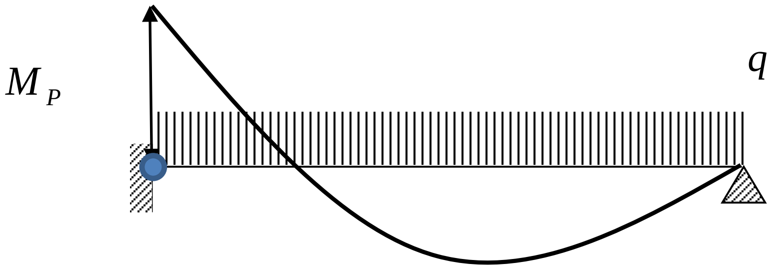
$$M_f = -\frac{q\ell^2}{8}\left(1 - \frac{x}{\ell}\right) + \frac{q\ell}{2}x - \frac{qx^2}{2}$$

$$\frac{dM_f}{dx} = \frac{ql}{8} + \frac{ql}{2} - qx = q\left(\frac{5\ell}{8} - x\right) \qquad \frac{dM_f}{dx} = 0 \quad x = \frac{5\ell}{8} \quad M_f\Big|_{x=\frac{5\ell}{8}} = \frac{9\,q\ell^2}{128} < \frac{q\ell^2}{8}$$

$$M_{máximo} = \frac{q\ell^2}{8} \quad en \quad x = 0 \qquad M_E = \frac{q\ell^2}{8} \quad \Rightarrow \quad q_{LE} = \frac{8M_E}{\ell^2}$$




$$M_P = \frac{q\ell^2}{8} \quad \Rightarrow \quad q_{LP} = \frac{8M_P}{\ell^2}$$




$$M_f = -M_P\left(1-\frac{x}{\ell}\right)+\frac{q\ell}{2}x-\frac{qx^2}{2}$$

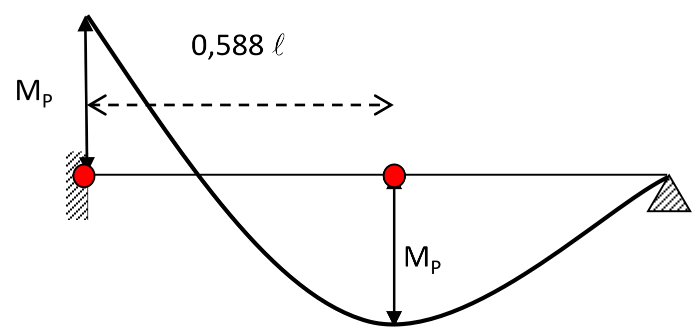
$$\frac{dM_f}{dx}=\frac{M_P}{\ell}+\frac{q\ell}{2}-qx \qquad \frac{dM_f}{dx}=0 \quad\Rightarrow\quad x=\ell\left(0{,}5+\frac{M_P}{q\ell^2}\right)=\ell\,(0{,}5+\lambda) \qquad \lambda=\frac{M_P}{q\ell^2}$$

$$M_f=\frac{q\ell^2}{2}\left[-2\lambda\,(1-0{,}5-\lambda)+(0{,}5+\lambda)-(0{,}5+\lambda)^2\right]=\frac{q\ell^2}{2}\left[\lambda^2-\lambda+0{,}25\right]$$

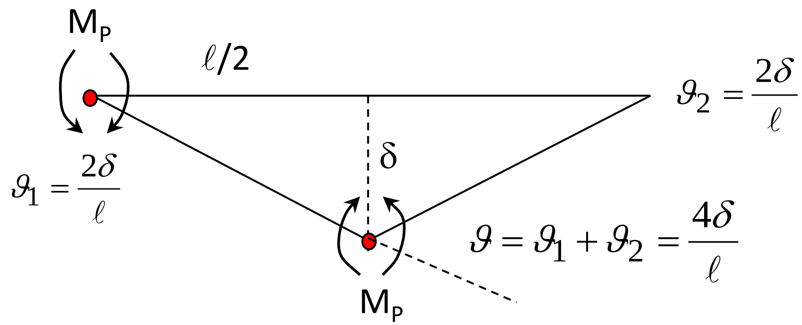
$$M_P=\lambda\,q\ell^2=\frac{q\ell^2}{2}\left[\lambda^2-\lambda+0{,}25\right] \quad\Rightarrow\quad \frac{q\ell^2}{2}\left[\lambda^2-3\lambda+0{,}25\right]=0$$

$$\lambda=0{,}0857864=\frac{1}{11{,}7} \qquad x=0{,}587864\,\ell$$

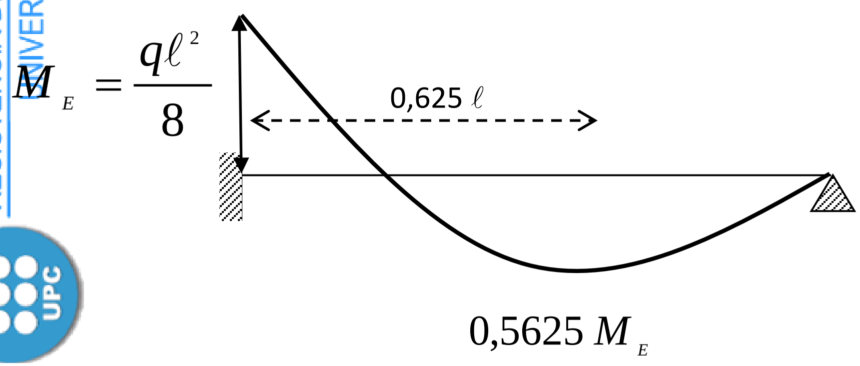
$$M_P=\frac{1}{11{,}7}q_{LP}\ell^2 \qquad q_{LP}=\frac{11{,}7\,M_P}{\ell^2}$$

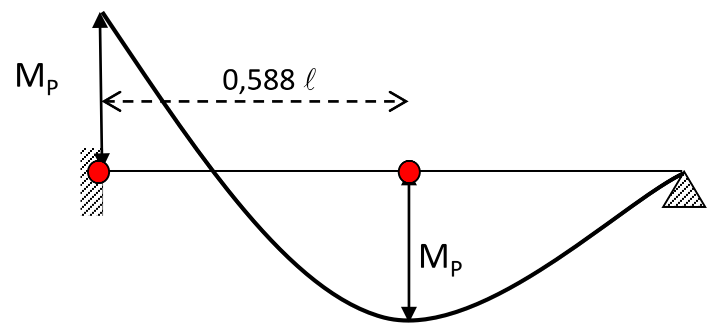



$$q_{LP}=\frac{12\,M_P}{\ell^2} \quad error\ 2{,}6\%$$

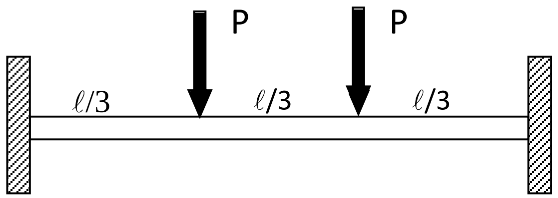



$$\frac{1}{2}\,\ell\,\delta\, q_{LP} = M_P\,\frac{2\delta}{\ell} + M_P\,\frac{4\delta}{\ell} \quad\Rightarrow\quad q_{LP} = \frac{12 M_P}{\ell^2}$$




$$\frac{q_{LP}}{q_{LE}} = \frac{\dfrac{11{,}7\ M_P}{\ell^2}}{\dfrac{8\ M_E}{\ell^2}} = 1{,}46\,\frac{M_P}{M_E} = 1{,}46 \times 1{,}5 = 2{,}2$$

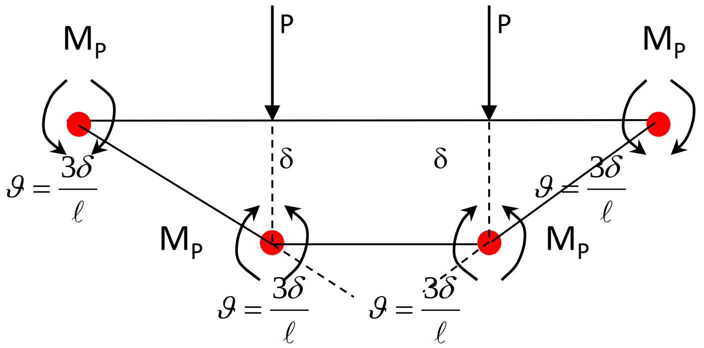



Hyperstatic degree 2

Critical Sections 4

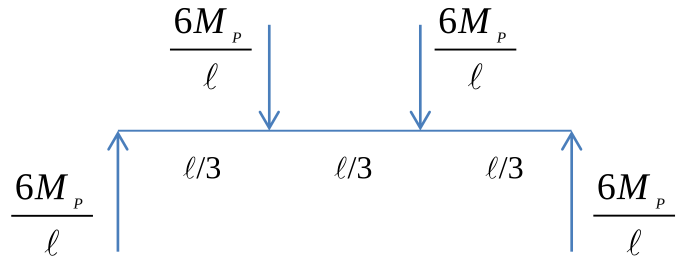
Plastic Hinges 3

**SYMMETRIC MECHANISM**




$$P\,\delta + P\,\delta = M_P\frac{3\delta}{\ell} + M_P\frac{3\delta}{\ell} + M_P\frac{3\delta}{\ell} + M_P\frac{3\delta}{\ell}$$

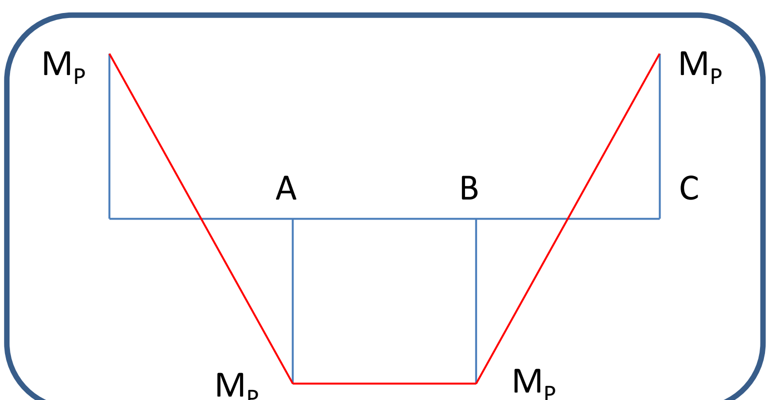
$$P_{LP} = \frac{6M_P}{\ell}$$

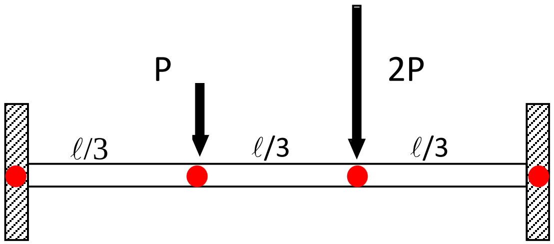



**FORCE BALANCE**

**BENDING MOMENT AT A B and C**

$$\begin{cases} M_A = -M_P + \frac{6M_P}{\ell} \times \frac{\ell}{3} = M_P \\ M_B = -M_P + \frac{6M_P}{\ell} \times \frac{2\ell}{3} - \frac{6M_P}{\ell} \times \frac{\ell}{3} = M_P \\ M_C = -M_P + \frac{6M_P}{\ell} \times \ell - \frac{6M_P}{\ell} \times \frac{2\ell}{3} - \frac{6M_P}{\ell} \times \frac{\ell}{3} = -M_P \end{cases}$$






Hyperstatic dgree 2

Critical Sections 4

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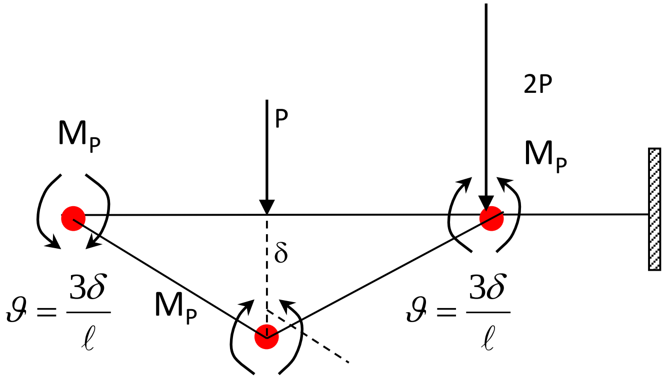
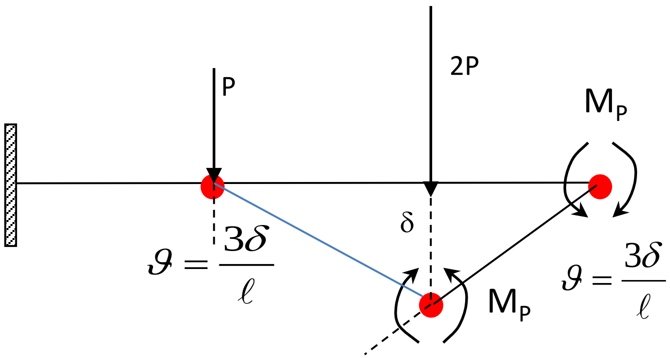


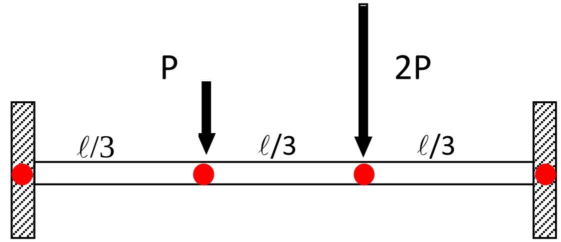

$$P\,\delta = M_P\frac{3\delta}{\ell} + M_P\left(\frac{3\delta}{\ell}+\frac{3\delta}{\ell}\right) + M_P\frac{3\delta}{\ell}$$

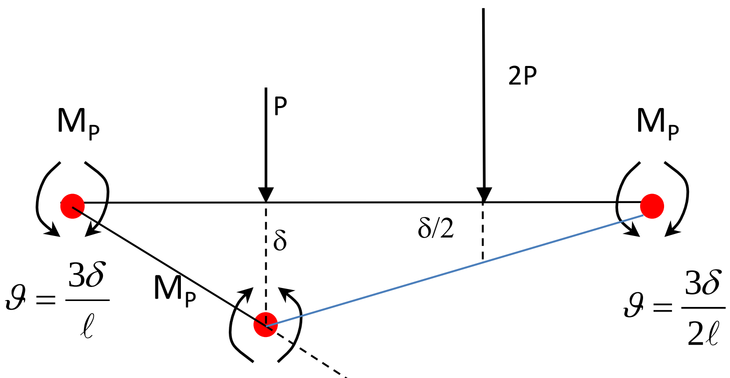
$$P_{LP} = \frac{12\,M_P}{\ell}$$




$$2\,P\,\delta = M_P\frac{3\delta}{\ell} + M_P\left(\frac{3\delta}{\ell}+\frac{3\delta}{\ell}\right) + M_P\frac{3\delta}{\ell}$$

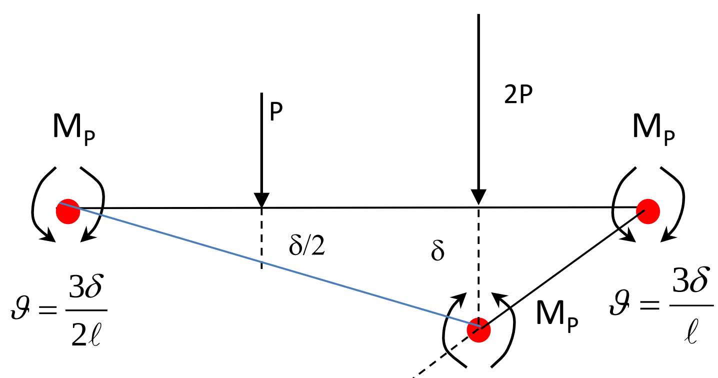
$$P_{LP} = \frac{6\,M_P}{\ell}$$



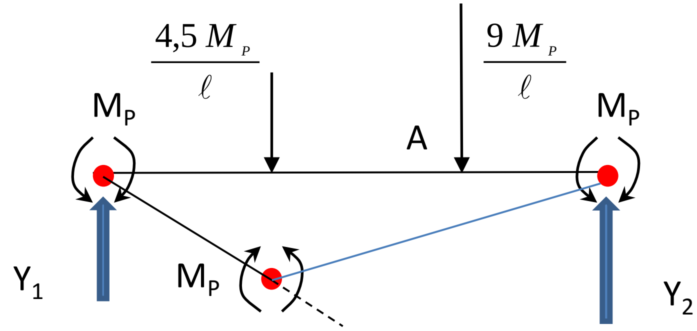

$$P\,\delta + 2P\,\frac{\delta}{2} = M_P\,\frac{3\delta}{\ell} + M_P\left(\frac{3\delta}{\ell} + \frac{3\delta}{2\ell}\right) + M_P\,\frac{3\delta}{2\ell}$$

$$P_{LP} = \frac{4{,}5\,M_P}{\ell}$$




$$P\,\frac{\delta}{2} + 2P\,\delta = M_P\,\frac{3\delta}{2\ell} + M_P\left(\frac{3\delta}{\ell} + \frac{3\delta}{2\ell}\right) + M_P\,\frac{3\delta}{\ell}$$

$$P_{LP} = \frac{3{,}6\,M_P}{\ell}$$



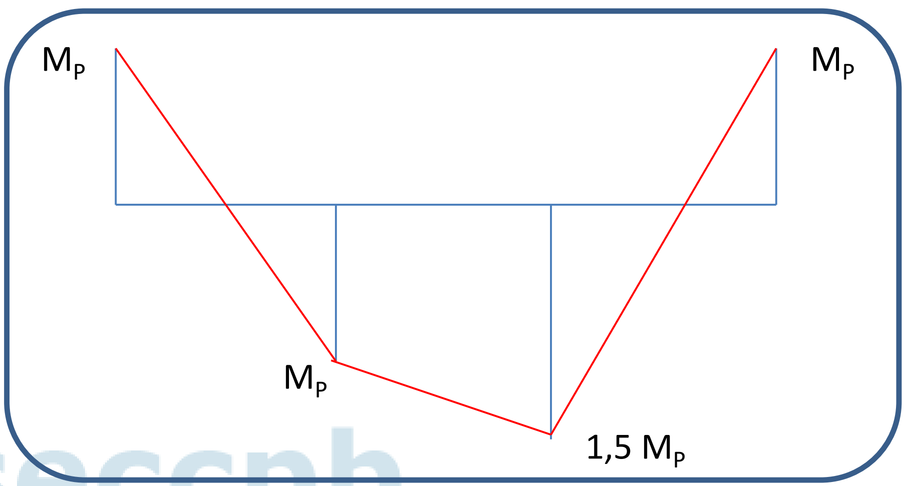

$$Y_1 + Y_2 = \frac{13,5\,M_P}{\ell}$$

$$Y_1 \times \frac{\ell}{3} - M_P = M_P \qquad Y_1 = \frac{6\,M_P}{\ell}$$

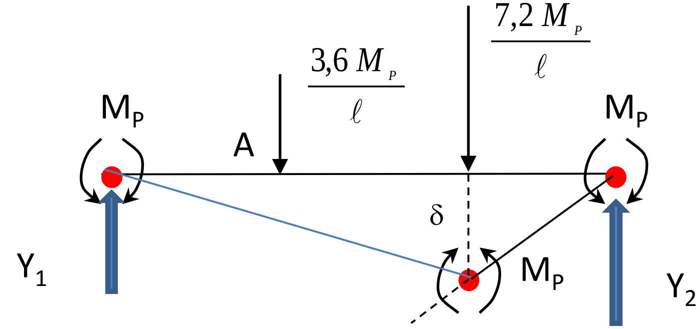
$$M_A = \frac{6\,M_P}{\ell} \times \frac{2\ell}{3} - M_P - \frac{4,5\,M_P}{\ell} \times \frac{\ell}{3} = 1,5\,M_P$$

$$Y_2 \times \frac{2\ell}{3} - M_P - \frac{9\,M_P}{\ell} \times \frac{\ell}{3} = M_P \qquad Y_1 = \frac{7,5\,M_P}{\ell}$$

$$M_A = -M_P + \frac{7,5\,M_P}{\ell} \times \frac{\ell}{3} = 1,5\,M_P$$




**The plastic bending moment is exceeded**



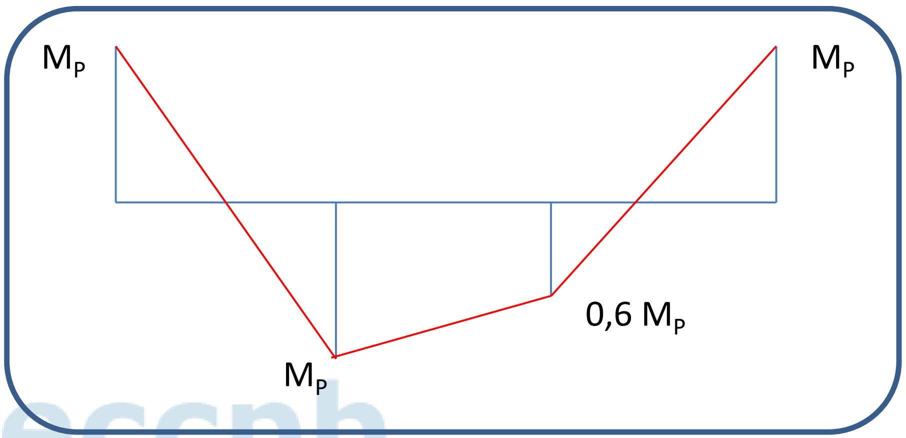

$$Y_1 + Y_2 = \frac{10{,}8\,M_P}{\ell}$$

$$Y_1 \times \frac{2\ell}{3} - M_P - \frac{3{,}6\,M_P}{\ell} \times \frac{\ell}{3} = M_P \qquad Y_1 = \frac{4{,}8\,M_P}{\ell}$$

$$M_A = -M_P + \frac{4{,}8\,M_P}{\ell} \times \frac{\ell}{3} = 0{,}6\,M_P$$

$$Y_2 \times \frac{\ell}{3} - M_P = M_P \qquad Y_1 = \frac{6\,M_P}{\ell}$$

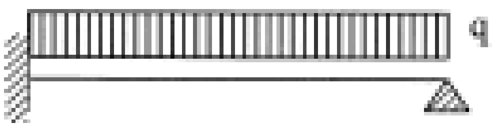
$$M_A = \frac{6\,M_P}{\ell} \times \frac{2\ell}{3} - M_P - \frac{7{,}2\,M_P}{\ell} \times \frac{\ell}{3} = 0{,}6\,M_P$$

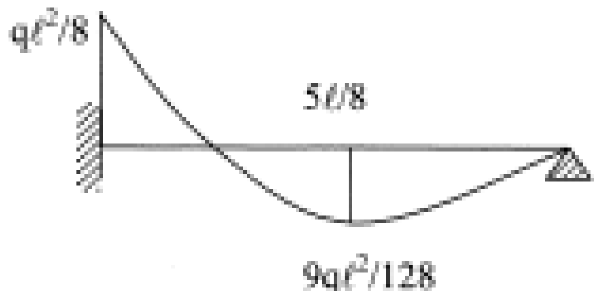



**$M_P$ is not exceded**

**Static Theorem, or Maximum Theorem**

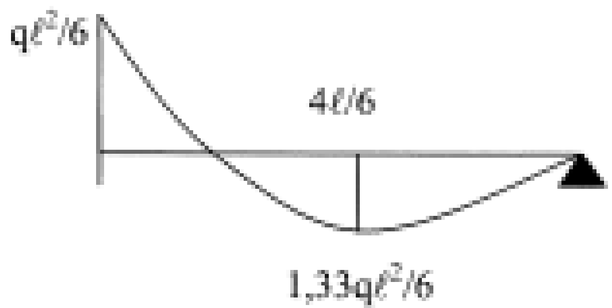
Any distribution of bending moments which is compatible with the equilibrium equations, in which the bending moment does not exceed the critical plastic moment Mp in any section, is caused by a load which is lesser or equal than the critical load.



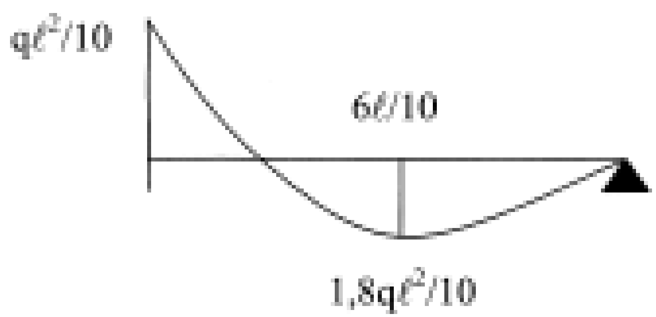

$q_{LP}=8M_P/\ell^2$

$4{,}51\ M_P/\ell^2$




$q^1{}_{LP}=6M_P/1{,}33\ell^2$

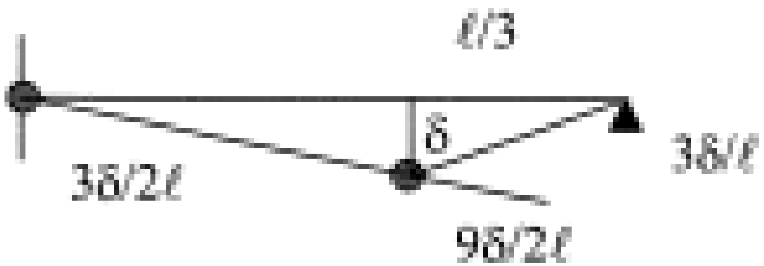
$5{,}55\ M_P/\ell^2$




$q^2{}_{LP}=10M_P/1{,}8\ell^2$

**Kinematic Theorem, or Minimum Theorem**

The critical load is the minimum of the critical loads obtained by placing plastic hinges in an arbitrary manner in several sections of the structural system

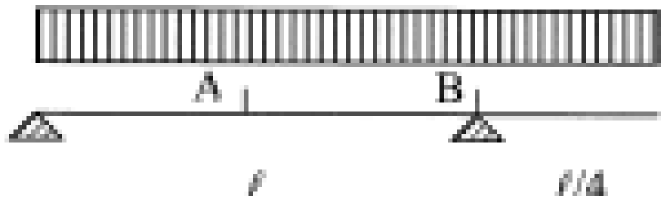



$$q\,0{,}5\,\ell\,\delta = M_P\,3\delta/2\ell + M_P\,9\delta/2\ell$$

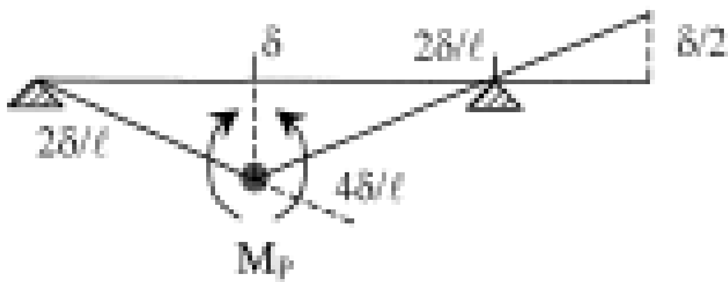
$$q'_{LP} = 12\,M_P/\ell^2 > 8\,M_P/\ell^2 = q_{LP}$$

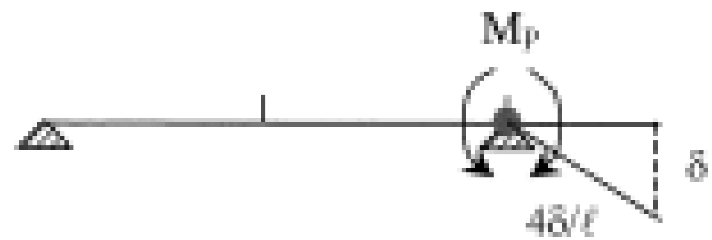
$$5{,}56\,M_P/\ell^2 < \mathbf{q_{LP} = 8\,M_P/\ell^2} < 12\,M_P/\ell^2$$

# Exercise 1



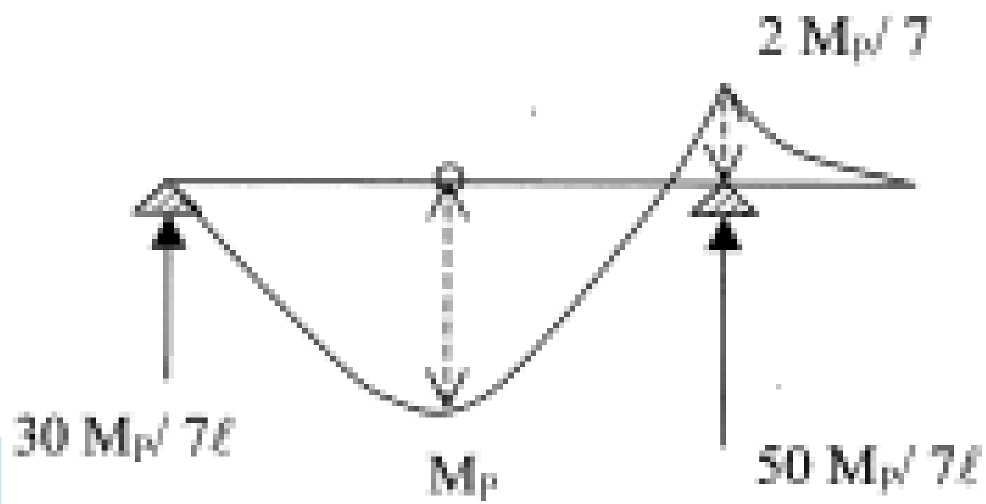

| | | |
|---|---|---|
| Critical Sections | A | B |
| Hyperestatic degree | 0 | |
| Plastic Hinges | 0 | |
| Mechanisms | 2 | |



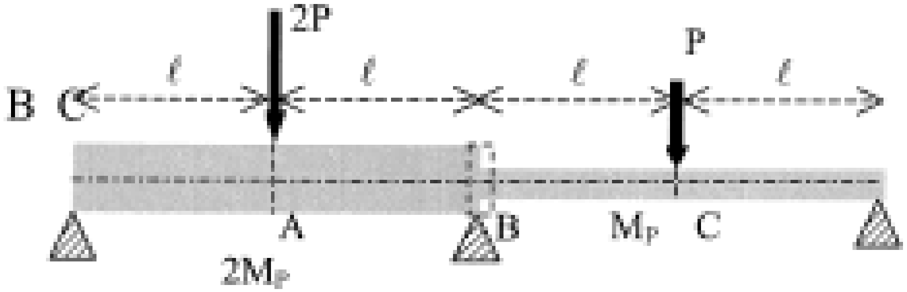




$$\frac{1}{2}\ell\delta q - \frac{1}{2}\frac{\ell}{4}\frac{\delta}{2}q = M_P\frac{4\delta}{\ell} \qquad q_{LP} = \frac{64 M_P}{7\ell^2}$$

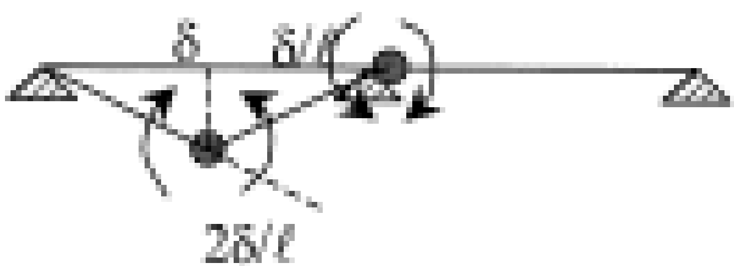
$$\frac{1}{2}\frac{\ell}{4}\delta q = M_P\frac{4\delta}{\ell} \qquad q_{LP} = \frac{32 M_P}{\ell^2}$$



# Exercise 2

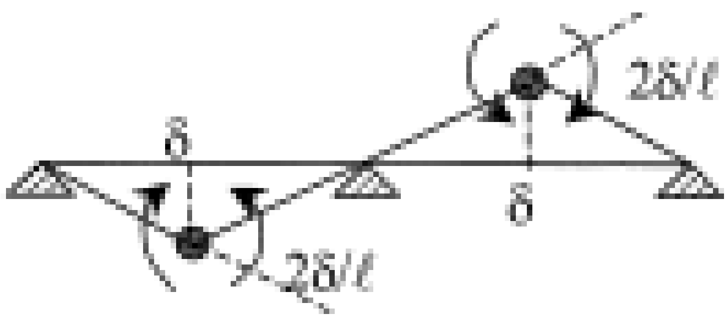



| | |
|---|---|
| Critical Sections | A |
| Hyperestatic degree | 1 |
| Plastic Hinges | 2 |
| Mechanisms | 3 |



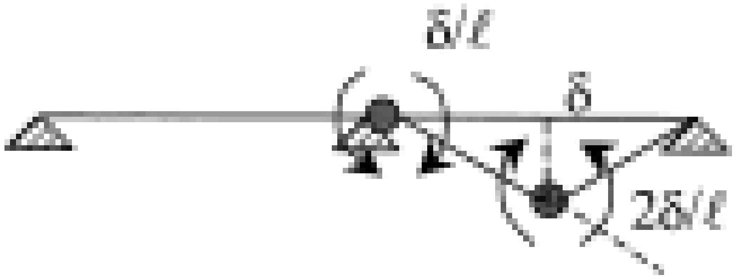

$$2P\ \delta = 2M_P \frac{2\delta}{\ell} + M_P \frac{\delta}{\ell}$$

$$P_{LP} = \frac{5M_P}{2\ell}$$



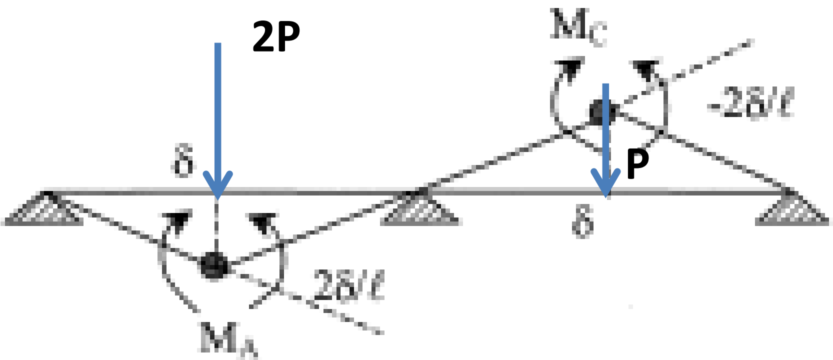

$$2P\,\delta - P\,\delta = 2M_P \frac{2\delta}{\ell} + M_P \frac{2\delta}{\ell}$$

$$P_{LP} = \frac{6M_P}{\ell}$$



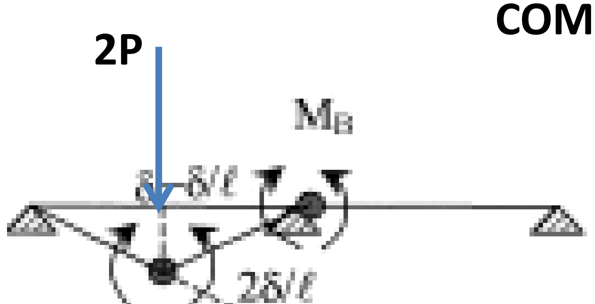

$$P\,\delta = M_P \frac{\delta}{\ell} + M_P \frac{2\,\delta}{\ell}$$

$$P_{LP} = \frac{3\,M_P}{\ell}$$

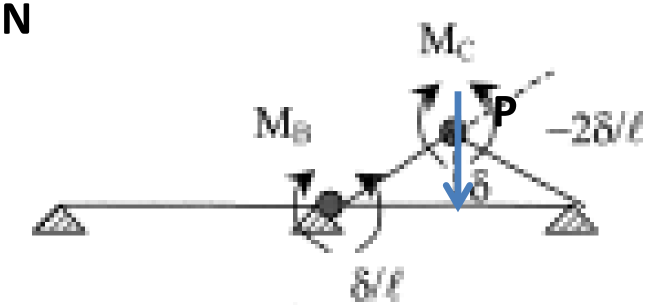



$$m_3 \rightarrow m_1 - m_2$$

**COMBINATION**



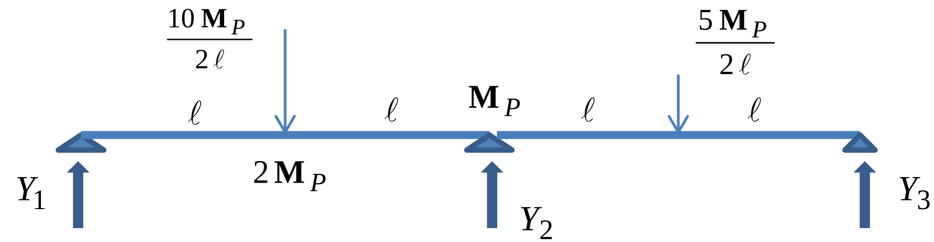

$+$




$$2Pl = 2\,M_A - M_B \qquad -Pl = -2\,M_C + M_B$$

$$P\,\ell = 2\,M_A - 2\,M_C$$

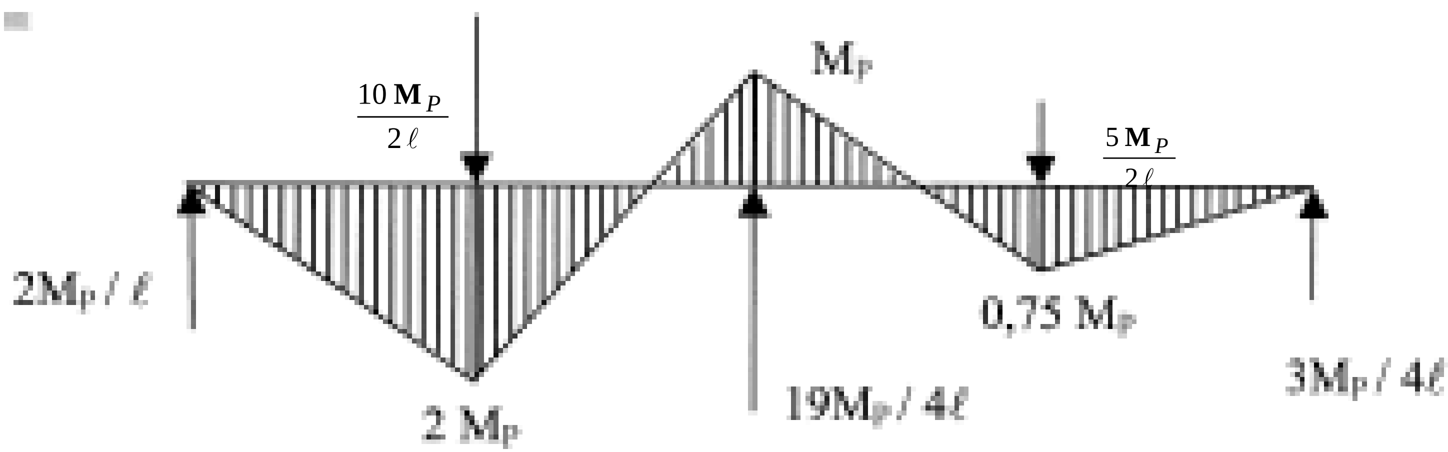
$$M_A = 2\,M_P \qquad M_B = -M_P \qquad P_{LP} = \frac{6\,M_P}{\ell}$$




$$Y_1 \times \ell = 2\,M_P \qquad Y_1 = \frac{2\,M_P}{\ell}$$

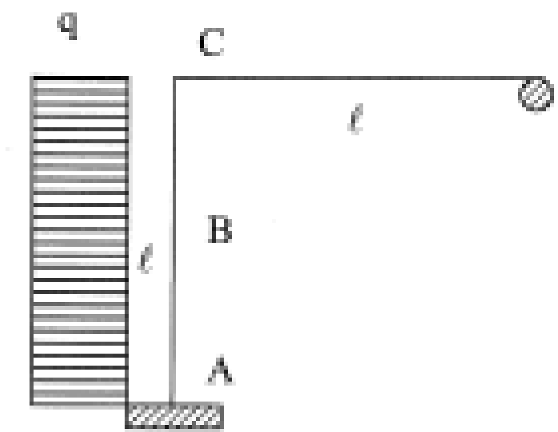
$$Y_3 \times 2\ell - \frac{5\,M_P}{2\,\ell} \times \ell = -M_P \qquad Y_1 = \frac{3\,M_P}{4\,\ell}$$

$$Y_2 = \frac{15\,M_P}{2\,\ell} - \frac{2\,M_P}{\ell} - \frac{3\,M_P}{4\,\ell} = \frac{19\,M_P}{4\,\ell}$$

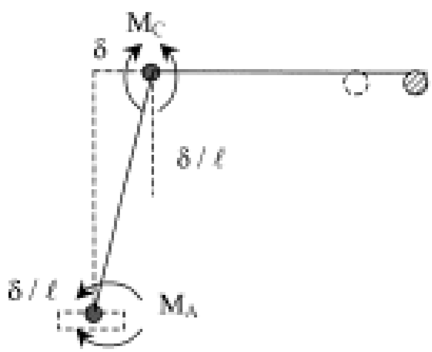



**The plastic bending moment $M_P$ is not exceeded**

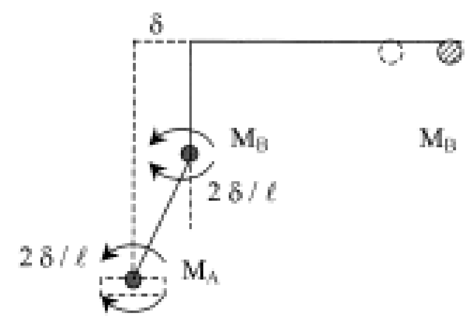
# Exercise 3



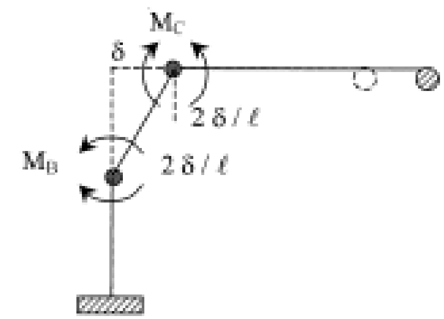

| | |
|---|---|
| Critical Sections | A B C |
| Hyperestatic degree | 1 |
| Plastic Hinges | 2 |
| Mechanisms | 3 |




Mecanismo 1




Mecanismo 2




Mecanismo 3

**Mecanismo 1**

$$q\frac{1}{2}\delta\,\ell = -M_A\frac{\delta}{\ell} + M_c\frac{\delta}{\ell}$$

$$-M_A = M_C = M_P$$

$$q_{LP} = \frac{4M_P}{\ell^2}$$

**Mecanismo 2**

$$q\left(\frac{1}{2}\delta\frac{\ell}{2} + \delta\frac{\ell}{2}\right) = -M_A\frac{2\delta}{\ell} + M_B\frac{2\delta}{\ell}$$

$$-M_A = M_B = M_P$$

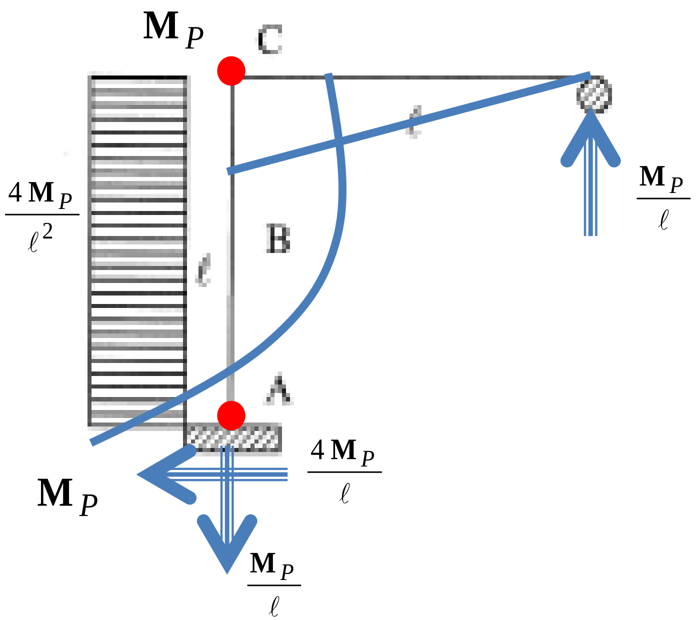
$$q_{LP} = \frac{16M_P}{3\ell^2}$$

**Mecanismo 3**

$$q\frac{1}{2}\delta\frac{\ell}{2} = -M_B\frac{2\delta}{\ell} + M_c\frac{2\delta}{\ell}$$

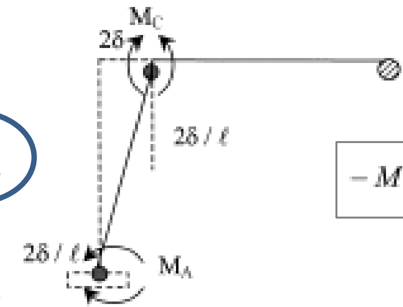
$$-M_B = M_C = M_P$$

$$q_{LP} = \frac{16M_P}{\ell^2}$$




m3 **COMBINATION $m_1+m_2$**

m1



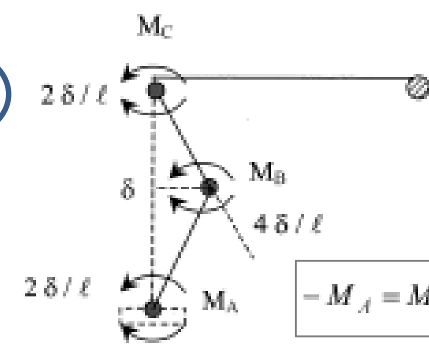

Mecanismo Traslacional

$$-M_A = M_C = M_P$$

$$q\frac{1}{2}\,2\delta\,\ell = -M_A\frac{2\delta}{\ell} + M_c\frac{2\delta}{\ell}$$

$$q_{LP} = \frac{4M_P}{\ell^2}$$

m2

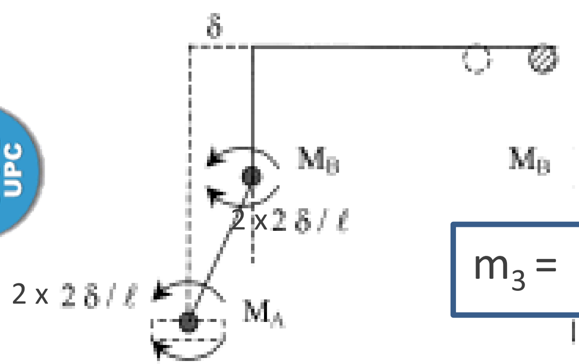



Mecanismo de Pilar

$$-M_A = M_B = -M_C = M_P$$

$$q\frac{1}{2}\delta\,\ell = -M_A\frac{2\delta}{\ell} + M_B\frac{4\delta}{\ell} - M_C\frac{2\delta}{\ell}$$

$$q_{LP} = \frac{16M_P}{\ell^2}$$




$m_3 = m_1 + m_2$

$$q\frac{3}{2}\delta\,\ell = -M_A\frac{4\delta}{\ell} + M_B\frac{4\delta}{\ell}$$

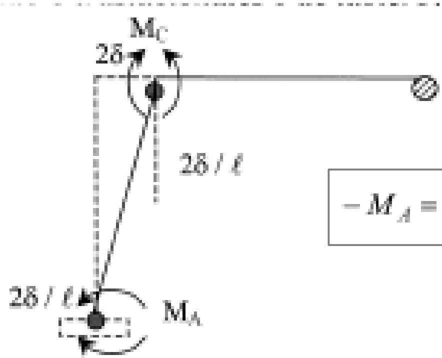
$$-M_A = M_B = M_P$$

$$q_{LP} = \frac{16M_P}{3\ell^2}$$

# COMBINATION $m_1+m_2$

$m_3 = m_1 + m_2$

$m_3$

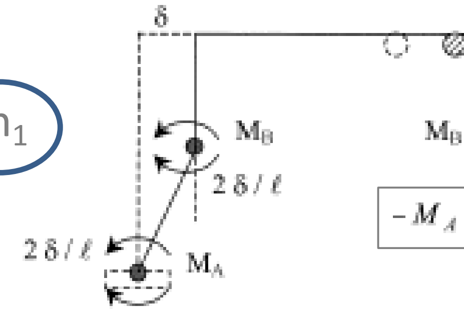



$-M_A = M_C = M_P$

$$q \frac{1}{2}\, 2\,\delta\,\ell = -M_A \frac{2\delta}{\ell} + M_c \frac{2\delta}{\ell}$$

$$q_{LP} = \frac{4\,M_P}{\ell^2}$$

$m_1$

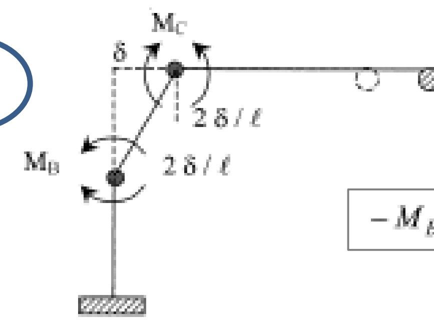



$-M_A = M_B = M_P$

$$q\left(\frac{1}{2}\delta\frac{\ell}{2} + \delta\frac{\ell}{2}\right) = -M_A\frac{2\delta}{\ell} + M_B\frac{2\delta}{\ell}$$

$$q_{LP} = \frac{16\,M_P}{3\,\ell^2}$$

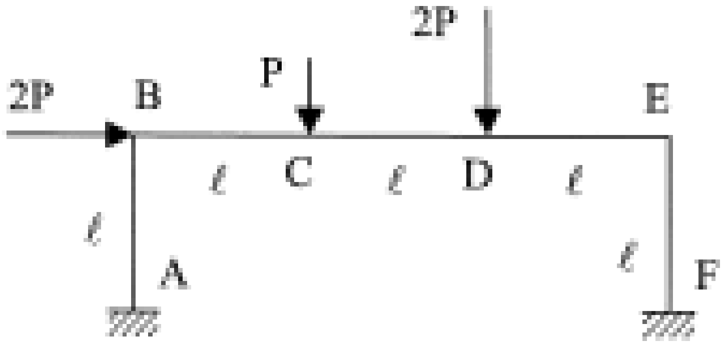
$m_2$



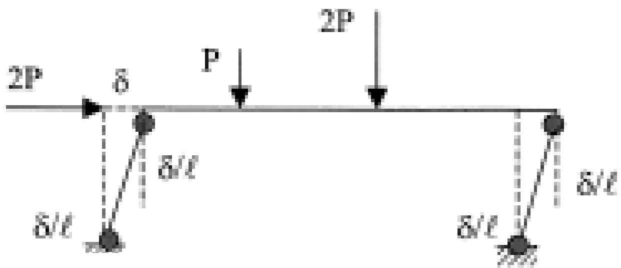

$-M_B = M_C = M_P$

$$q\frac{1}{2}\delta\frac{\ell}{2} = -M_B\frac{2\delta}{\ell} + M_c\frac{2\delta}{\ell}$$

$$q_{LP} = \frac{16\,M_P}{\ell^2}$$



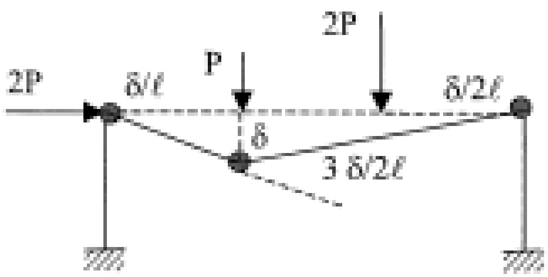

| | |
|---|---|
| Critical Sections | A B C D E F |
| Hyperestatic degree | 3 |
| Plastic Hinges | 4 |
| Mechanisms | 3 |




$-M_A = M_B = -M_E = M_F = M_P$

$$2P\delta = -M_A\frac{\delta}{\ell} + M_B\frac{\delta}{\ell} - M_E\frac{\delta}{\ell} + M_F\frac{\delta}{\ell}$$

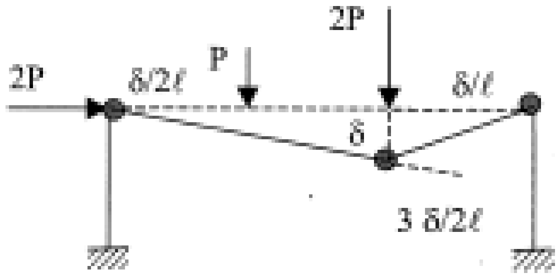
$$P_{LP} = \frac{2M_P}{\ell}$$




$$P\delta + 2P\frac{\delta}{2} = -M_B\frac{\delta}{\ell} + M_C\frac{3\delta}{2\ell} - M_E\frac{\delta}{2\ell}$$

$-M_B = M_C = -M_E = M_P$

$$P_{LP} = \frac{6M_P}{4\ell}$$




$$-M_B = M_D = -M_E = M_P$$

$$P\frac{\delta}{2} + 2P\,\delta = -M_B\frac{\delta}{2\ell} + M_D\frac{3\delta}{2\ell} - M_E\frac{\delta}{\ell}$$

$$P_{LP} = \frac{6M_P}{5\ell}$$

| Mechanism | Load | $M_A$ | $M_B$ | $M_C$ | $M_D$ | $M_E$ | $M_F$ | $P_{LP}$ |
|---|---|---|---|---|---|---|---|---|
| Translation al | $2P\ell$ | -1 | +1 | 0 | 0 | -1 | +1 | $2\,M_p/\ell$ |
| Lintel 1 | $4P\ell$ | 0 | -2 | +3 | 0 | -1 | 0 | $1{,}5\,M_p/\ell$ |
| Lintel 2 | $5P\ell$ | 0 | -1 | 0 | +3 | -2 | 0 | $1{,}2\,M_p/\ell$ |
| $C_1$ $2T+D_1$ | $8P\ell$ | -2 | 0 | +3 | 0 | -3 | +2 | $1{,}25\,M_p/\ell$ |
| $C_2$ $T+D_2$ | $7P\ell$ | -1 | 0 | 0 | +3 | -3 | +1 | $1{,}14\,M_p/\ell$ |

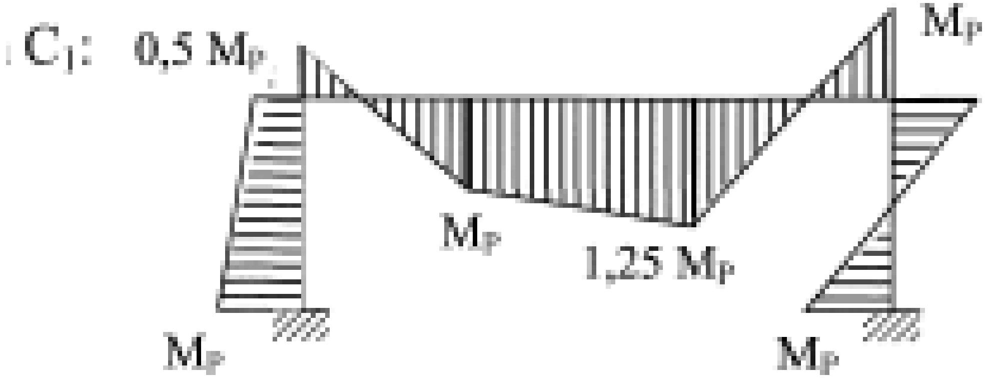
$C_3$ $D_1 + D_2$ $P_{LP} = 12\,M_P / 9\,\ell = 1{,}33\,M_P/\ell$

$1{,}5\,M_P/\ell > 1{,}33\,M_P/\ell > 1{,}2\,M_P/\ell$

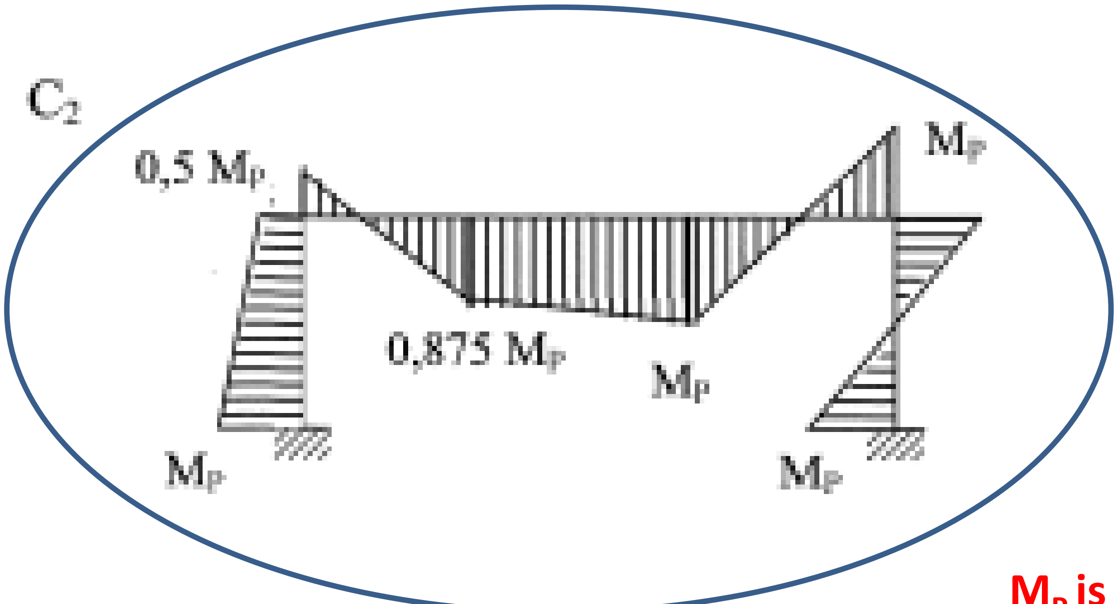
$C_2$ $T + D_2$ $P_{LP} = 8\,M_P / 7\,\ell = 1{,}14\,M_P/\ell$

$1{,}14\,M_P/\ell < 1{,}2\,M_P/\ell < 2\,M_P/\ell$

Escola




$M_P$ is exceeded




$M_P$ is not reached